Nowhere-zero 3-flows in arc-transitive graphs on solvable groups

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circulations

Definition

Let $D = (V(D), A(D))$ be a digraph and A an abelian group. A circulation in D over A is a function

$$f : A(D) \to A$$

such that

$$\sum_{a \in A^+(v)} f(a) = \sum_{a \in A^-(v)} f(a), \quad \text{for all } v \in V(D),$$

where $A^+(v)$ ($A^-(v)$, respectively) is the set of arcs of D leaving from v (entering into v, respectively).

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where $A^+(v)$ ($A^-(v)$, respectively) is the set of arcs of D leaving from v (entering into v, respectively).

We say that f is nowhere-zero if $f(a) \neq 0$ for every $a \in A(D)$, where 0 is the identity element of A.

Theorem

(W. Tutte 1954)
A plane digraph is k-face-colorable if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

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Four-Color-Theorem Restated:
Every planar graph admits a nowhere-zero circulation over $\mathbb{Z}_4$.

integer flows

Definition

A nowhere-zero circulation f over $\mathbb{Z}$ in a digraph D is called a (nowhere-zero) k-flow if

$$-(k-1) \leq f(a) \leq k-1, \quad \text{for all } a \in A(D)$$

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A graph admits a k-flow if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

Four-Color-Theorem Again:
Every planar graph admits a 4-flow.

Theorem

A graph admits a 2-flow if and only if its vertices all have even degrees.

Theorem

A 2-edge-connected cubic graph admits a 3-flow if and only if it is bipartite.

Tutte's 5-flow conjecture

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(The 6-flow theorem, P. Seymour 1981)
Every 2-edge-connected graph admits a 6-flow.

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Theorem

(M. Kochol 2001)
The 3-flow conjecture is true if and only if every 5-edge-connected graph admits a 3-flow.

recent breakthrough

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(C. Thomassen 2012)
Every 8-edge-connected graph admits a 3-flow.

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Theorem

(L. M. Lovász, C. Thomassen, Y. Wu and C. Q. Zhang 2013)
Every 6-edge-connected graph admits a 3-flow.

motivation

A graph Γ is G-vertex-transitive if $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set of vertices of Γ.
Γ is vertex-transitive if it is $\mathrm{Aut}(\Gamma)$-vertex-transitive.

Theorem

(M. E. Watkins 1969; W. Mader 1970)
Every vertex-transitive graph of valency d is d-edge-connected.

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Conjecture

(Vertex-transitive version of the 3-flow conjecture)
Every vertex-transitive graph of valency at least 4 admits a 3-flow.

It suffices to prove this for vertex-transitive graphs of valency 5.

3-flows in Cayley graphs on nilpotent groups

Theorem

(P. Potačnik 2005)
Every Cayley graph of valency at least 4 *on a finite abelian group admits a 3-flow.*

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Theorem
(M. Nánásiová and M. Škoviera 2009)
Every Cayley graph of valency at least 4 on a finite nilpotent group admits a 3-flow.

A finite group is nilpotent if it is the direct product of its Sylow subgroups.

an intermediate goal

Prove that every graph of valency at least 4 admitting a solvable vertex-transitive group of automorphisms admits a 3-flow.

As before it suffices to prove this for the case of valency 5.

symmetry of graphs

Definition

Let Γ be a graph admitting a group G as a group of automorphisms.

If G is transitive on the set of vertices (edges, respectively) of Γ, then Γ is called G-vertex-transitive (G-edge-transitive, respectively).

Γ is G-arc-transitive if it is G-vertex-transitive and G is transitive on the set of arcs of Γ, where an arc is an ordered pair of adjacent vertices.

result so far

Theorem

(X. Li and S. Zhou 2013-14)
Let G be a finite solvable group. Then every G-arc-transitive graph with valency at least 4 admits a 3-flow.

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Let G be a finite solvable group. Then every G-arc-transitive graph with valency at least 4 admits a 3-flow.

- Any G-arc-transitive graph is G-vertex-transitive and G-edge-transitive
- Any G-vertex-transitive and G-edge-transitive graph with odd valency is G-arc-transitive

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Therefore, our result is equivalent to:

Theorem

Let G be a finite solvable group. Then every G-vertex-transitive and G-edge-transitive graph with valency at least 4 admits a 3-flow.

solvable groups

Definition

$G' := [G, G]$: derived subgroup of G, the subgroup of G generated by all commutators $x^{-1}y^{-1}xy$, $x, y \in G$

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The least integer n with $G^{(n)} = 1$ is the derived length of G.

- Solvable groups with derived length 1 are precisely nontrivial abelian groups.
- Subgroups and quotient groups of a solvable group are solvable.
- Any solvable group G contains a normal abelian subgroup N such that G/N has a smaller derived length.

multicovers

Definition

Let Γ be a graph and $\mathcal{P}$ a partition of $V(\Gamma)$.

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Γ is a multicover of the quotient $\Gamma_{\mathcal{P}}$ if for each pair of adjacent $P, Q \in \mathcal{P}$, the subgraph $\Gamma[P, Q]$ of Γ induced by $P \cup Q$ is a t-regular bipartite graph with bipartition $\{P, Q\}$ for some integer $t \geq 1$ independent of P, Q.

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Lemma

Let $k \geq 2$ be an integer. If a graph admits a k-flow, then its multicovers all admit a k-flow.

normal quotients

Definition

Let Γ be a G-vertex-transitive graph, and let $N \trianglelefteq G$.

The set $\mathcal{P}_N$ of N-orbits on $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$, called a G-normal partition of $V(\Gamma)$.

Denote $\Gamma_N := \Gamma_{\mathcal{P}_N}$.

Lemma

Let Γ be a connected G-vertex-transitive graph. Let $N \trianglelefteq G$ be intransitive on $V(\Gamma)$. Then

(a) *Γ_N is G/N-vertex-transitive under the induced action of G/N on $\mathcal{P}_N$;*

(b) *for $P, Q \in \mathcal{P}_N$ adjacent in Γ_N, $\Gamma[P, Q]$ is a regular subgraph of Γ;*

(c) *if in addition Γ is G-arc-transitive, then Γ_N is G/N-arc-transitive and Γ is a multicover of Γ_N.*

result so far

Theorem

(X. Li and S. Zhou 2013-14)
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Let G be a finite solvable group. Then every G-arc-transitive graph with valency at least 4 admits a 3-flow.

- If $\texttt{val} = 4$, then the graph has a 2-flow and hence a 3-flow.
- If $\texttt{val} \geq 6$, then the graph is 6-edge-connected and so admits a 3-flow by LTWZ (2013).
- It is boiled down to the case $\texttt{val} = 5$.

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We prove:

Claim

If G is solvable, then every G-arc-transitive graph with valency at least 4 and not divisible by 3 admits a 3-flow.

outline of proof

- We may assume G is faithful on $V(\Gamma)$. We may also assume the graphs under consideration are connected.

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- Let G be a finite solvable group with derived length $n(G) = n + 1$.

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- Let G be a finite solvable group with derived length $n(G) = n + 1$.
- Let Γ be a connected G-arc-transitive graph such that $\mathtt{val}(\Gamma) \geq 4$ and $\mathtt{val}(\Gamma)$ is not divisible by 3.

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- If $\mathtt{val}(\Gamma)$ is even, Γ has a 2-flow and so a 3-flow.
- Assume $\mathtt{val}(\Gamma) \geq 5$ is odd.

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- Then Γ_N is a connected G/N-arc-transitive graph, and Γ is a multicover of Γ_N.

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- $\texttt{val}(\Gamma_N)$ is a divisor of $\texttt{val}(\Gamma)$ and so is not divisible by 3.
- If $\texttt{val}(\Gamma_N) = 1$, then Γ is a regular bipartite graph of valency at least two and so admits a 3-flow.

- Assume $\mathtt{val}(\Gamma_N) > 1$.

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- Since Γ is a multicover of Γ_N, Γ admits a 3-flow.
- This completes the proof.

difficulty for vertex- but not arc-transitive graphs

A G-vertex- but not G-arc-transitive graph Γ may not be a multicover of its normal quotients Γ_N.

In fact, in this case blocks of a normal partition are not necessarily independent sets.

This makes a similar induction difficult.

a conjecture on Cayley graphs

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Theorem

(Alspach, Liu and Zhang 1996)
The conjecture above is true for cubic Cayley graphs on finite solvable groups.

Theorem

(Nedela and Škoviera 2001)
Any counterexample must be a regular cover over a Cayley graph on an almost simple group.

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Theorem

(Potačnik 2004)
Every connected cubic graph admitting a solvable vertex-transitive group of automorphisms admits a 4-flow or is isomorphic to the Petersen graph.

what we may do

One may try to prove:

Every Cayley graph of valency at least four on a finite solvable group admits a 3-flow.

This will generalize both [Alspach, Liu and Zhang 1996] and [Nánásiová and Škoviera 2009].

thank you for your attention